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# On One Solution of The Problem of Vibrations of Mechanical Systems with Moving Boundaries

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## Abstract

An analytical method for solving the wave equation describing the oscillations of systems with moving boundaries is considered. By changing the variables that stop the boundaries and leave the equation invariant, the original boundary value problem is reduced to a system of functional-difference equations, which can be solved using direct and inverse methods. An inverse method is described that makes it possible to approximate quite diverse laws of boundary motion by laws obtained from solving the inverse problem. New particular solutions are obtained for a fairly wide range of laws of boundary motion. A direct asymptotic method for the approximate solution of a functional equation is considered. An estimate of the errors of the approximate method was made depending on the speed of the boundary movement.

**Keywords:** Wave Equation, Boundary Value Problems, Oscillations of Systems with Moving Boundaries, Change of Variables, Laws of Motion of Boundaries, Functional Equations.

## 1. Introduction

One-dimensional systems, the boundaries of which move, are widely used in engineering: ropes of lifting installations [1, 4, 8, 11–15, 21], flexible gear links [1, 2, 5, 16, 19, 20], solid fuel rods [22], drill strings [8], etc. The presence of moving boundaries causes significant difficulties in describ-ing such systems; therefore, here, approximate methods of solution are mainly used [1–4, 8, 10, 16, 17, 20–22, 25–28]. Of the analytical methods, the most effective is the method proposed in [5], which consists in the selection of new variables that stop the boundaries and leave the wave equa-tion invariant. In [6], the solution is sought in the form of a superposition of two waves running towards each other. The method used in [7] is also effective, which consists in replacing the geo-metric variable with a purely imaginary variable, which makes it possible to reduce the wave equation to the Laplace equation and apply the method of the theory of functions of a complex variable to the solution.

This article considers an analytical method for solving the wave equation that describes the oscil-lations of systems with moving boundaries. By changing the variables that stop the boundaries and leave the equation invariant, the original boundary value problem is reduced to a system of func-tional-difference equations that can be solved using direct and inverse methods. An inverse meth-od is described that makes it possible to approximate quite diverse laws of boundary motion by laws obtained from solving the inverse problem. New particular solutions are obtained for a fairly wide range of laws of motion of boundaries. A direct asymptotic method for the approximate solu-tion of a functional equation is considered. An estimate of the errors of the approximate method is made depending on the speed of the boundary movement. This approach successfully combines the technique used in [5, 6, 9, 18, 23, 24].

### 1.1. Statement of the Problem.

Let us consider free oscillations in a system with moving boundaries.

$$u_{tt}(x,t) - a^2 u_{xx}(x,t) = 0. \qquad (1)$$

The boundary conditions at the fixed ends have the form

$$u\left(l_1(t),t\right) = 0;\ \ u\left(l_2(t),t\right) = 0. \qquad (2)$$

Here, $u(x,t)$– is the displacement of the point of the object with the coordinate x at time t; velocity of wave propagation in the system; $l_1(x)$, $l_2(x)$ - the laws of movement of borders. In the works [5, 6] Vesnitsky A.I. a fairly general method for selecting new variables for the wave equation was proposed. Following this method, the change of variables is made in the following form:

$$\xi = \varphi(t + x/a) - \psi(t - x/a);$$
$$\tau = a^{-1}\left[\varphi(t + x/a) + \psi(t - x/a)\right], \tag{3}$$

where $\varphi$ "and " $\psi$– are some functions. As a result of such a replacement, the original equation remains invariant (wave), and φ,ψ are determined from the condition of constancy ξ at the boundaries.

In new variables ξ, τ, defined by relation (3), the original problem (1) – (2) is reduced to the following

$$U_{\tau\tau}(\xi,\tau) - U_{\xi\xi}(\xi,\tau) = 0 \tag{4}$$

under boundary conditions

$$U(\ell_1(\tau),\tau) = 0;\ U_\xi(\ell_2(\tau),\tau) = 0;$$
$$(\ell_1(\tau) \le \xi \le \ell_2(\tau)). \tag{5}$$

Here $\tau$, $\xi$ dimensionless time (τ≥0) and dimensionless spatial coordinate; U(ξ,τ)=u(x,t) ; $l_i$ (τ) - the laws of movement of borders.

Boundary conditions (5) in variables ξ, τ are set on new, generally speaking, moving boundaries, the position of which depends on two functions φ "and " ψ. Since they φ "and " ψ are arbitrary, one can require that the boundary conditions are written on fixed boundaries, i.e., $l_1$=const and $l_2$=const ($l_2 > l_1$).

For this, it is φ "and " ψ necessary to satisfy the system of functional equations:

$$\begin{cases} \varphi(\tau + \ell_1(\tau)) - \psi(\tau - \ell_1(\tau)) = \ell_1; \\ \varphi(\tau + \ell_2(\tau)) - \psi(\tau - \ell_2(\tau)) = \ell_2, \end{cases} \tag{6}$$

which uniquely determine the functions φ "and " ψ through the known laws of boundary motion. When the borders move at a speed greater than the speed of wave propagation, the solution of the wave equation becomes incorrect, therefore, a restriction is imposed on the speed of the borders | $l'_i$ (τ)|<1. Constants $l_i$ can be arbitrary, but not equal values (for example, $l_1$=0, $l_2$=1). Then system (6) will take the form:

$$\begin{cases} \varphi(\tau + \ell_1(\tau)) = \psi(\tau - \ell_1(\tau)); \\ \varphi(\tau + \ell_2(\tau)) = \psi(\tau - \ell_2(\tau)) + 1, \end{cases} \tag{7}$$

The existence of a solution to this system was proved in [5].

Solution (4) - (5) is found by the Fourier method [24]:

$$U(\xi,\tau) = \sum_{n=1}^{\infty} \sin\left(\omega_{0n}\xi\right)\left(D_n \cos\left(\omega_{0n}\tau\right) + E_n \sin\left(\omega_{0n}\tau\right)\right) =$$
$$= \sum_{n=1}^{\infty} r_n \left\{\sin\left(\omega_{0n}(\tau+\xi) + \alpha_n\right) - \sin\left(\omega_{0n}(\tau-\xi) + \alpha_n\right)\right\},$$

where

$$\omega_{0n}(\varepsilon_0\tau) = \frac{\pi n}{\ell_2 - \ell_1};\ r_n = \frac{1}{2}\sqrt{D_n^2 + E_n^2};\ \alpha_n = arctg\left(E_n / D_n\right).$$

The solution obtained in [1–6, 8–10] has a form similar to (8).

Returning to the variables x and t, we get

$$u(x,t) = \sum_{n=1}^{\infty} r_n \left\{\sin\left(\omega_{0n}\varphi(t+x) + \alpha_n\right) - \sin\left(\omega_{0n}\psi(t-x) + \alpha_n\right)\right\}.$$

Here φ "and " ψ, they are found from the solutions of the system of functional equations (7) according to the known laws of motion of the boundaries, and the constants $D_n$, $E_n$ are determined from the initial conditions.

Generally speaking, it is not easy to solve system (7). There are two different approaches to solving it:

- is the inverse problem [5, 6, 8, 9, 18, 22, 23], i.e., according to the given "phases" of natural oscillations φ "and " ψ, finding the laws of motion of the boundaries $l_i$ (τ);
- is the direct problem [17, 22], i.e., finding the "phases" of natural oscillations according to the given laws of motion of the boundaries $l_i$ (τ).

### 1.2 Method Solution of the Inverse Problem.

To solve system (7) A.I. Vesnitsky [5] used the inverse method, i.e., given from the resulting system of equations, the laws of motion of the boundaries $l_1$ (τ) "and" $l_2$ (τ) are found. When solving the inverse problem, the equations of system (7) are reduced to the study of algebraic or transcendental equations with respect to $l_i$ (τ), which in many cases admit exact solutions. Based on the inverse problem Vesnitsky A.I. and Potapov A.I. [5, 6] obtained solutions for a fairly wide range of laws of boundary motion.

System (7) has infinitely many solutions, since on the interval [0, 1] the function φ(z) and on the interval [–1, 0] the function ψ(z) can be set arbitrarily, and using the method of successive approximations [24], the values of functions in other areas are found. It is enough for us to find one particular solution that determines the one-to-one correspondence of points z and points $y_1 = \varphi(z);\ y_2 = \psi(z).$ Of all the solutions, we are only interested in monotonic ones, and monotonic solutions in the case of boundary movement at a speed lower than the wave propagation speed ($|l_1'$ (τ)|<1 ; $|l_2'$ (τ)|<1) can only be monotonically increasing.

**Lemma.** If a function φ(z) – is monotonically increasing (decreasing), then the function ψ(z) – is also monotonically increasing (decreasing).

**Proof.** Indeed, from the first equation of system (7) at $\tau = \tau_0$ $\varphi(\tau_0 + \ell_1(\tau_0)) = \psi(\tau_0 - \ell_1(\tau_0)).$

Now suppose that $\tau_1 > \tau_2$ the function φ(z) also increases (decreases), then in the case of boundaries moving at a speed lower than the wave propagation speed ($|l_1'(\tau)|<1$ ; $|l_2'(\tau)|<1$), we will have:

$$\tau_1 + \ell_1(\tau_1) > \tau_0 + \ell_1(\tau_0);$$
$$\tau_1 - \ell_1(\tau_1) > \tau_0 - \ell_1(\tau_0).$$

Since the function φ(z) in this case increases (decreases), then in order to fulfill the first equality of system (7) at $\tau = \tau_1$, it is necessary that the function ψ(z) increases (decreases), i.e., the function ψ(z) is also increasing (decreasing).

Let us also show that the monotonic solution of system (7) in the case of boundary motion at a speed lower than the wave propagation velocity can only be increasing.

Indeed, given the inequality $\ell_1(\tau) < \ell_2(\tau)$, we get:

$$\tau + \ell_1(\tau) < \tau + \ell_2(\tau);\ \tau - \ell_1(\tau) > \tau - \ell_2(\tau).$$

Suppose that φ(z) "and " ψ(z) they decrease, then we can write:

$$\varphi(\tau + \ell_2(\tau)) < \varphi(\tau + \ell_1(\tau)) = \psi(\tau - \ell_1(\tau)) < \psi(\tau - \ell_2(\tau)).$$

However, this inequality contradicts the second equation of system (7). Therefore, functions φ(z) "and " ψ(z) can only be monotonically increasing. The lemma is proved. Note that from system (7) the functions φ(z) "and " ψ(z) are determined up to a constant in the sense that if φ(z) "and " ψ(z) the solution to system (7) is φ(z)+C " and " ψ(z)+C also a solution (here C-an arbitrary constant). Therefore, for definiteness, we can choose such a function ψ(z), that ψ(-1)=-1. At the same time, from the second equation of system (7) for τ=0, it follows that φ(1) = 0 From the first equation of system (7) for τ=0, we obtain φ(0)=ψ(0).

When assigning functions φ(z) "and " ψ(z), several arbitrary constants are introduced into them. The dependence of the laws of motion $l_1(\tau)$" and " $l_2(\tau)$ found on the values of these constants makes it possible to approximate quite diverse laws of motion of the boundaries by laws obtained from solving the inverse problem.

The set of inverse solutions is quite wide. The solutions below satisfy the relations:

$$\ell_1(0) = 0;\ \ell_2(0) = 1;\ \psi(-1) = -1.$$

The set of obtained laws of motion of boundaries is divided into classes:

1. The solutions shown in Table 1 are class A when the left boundary is fixed and φ(z)=ψ(z). Solutions numbered 1, 2, 3, 6 were obtained by A.I. Vesnitsky and A.I. Potapov [5, 6], solutions 4, 5, 7 were obtained for the first time.

**Table 1: Class A decisions**

| | $l_2(\tau)$ | $\varphi(z) = \psi(z)$ |
|---|---|---|
| 1 | $v\tau + 1$ | $\dfrac{Ln\left[(vz+1)/(1-v)\right]}{Ln\left[(1+v)/(1-v)\right]} - 1$ |
| 2 | $\sqrt{B\tau + B^2}\,/\,\lvert B\rvert$ | $\sqrt{Bz + B + 0,25} -$ <br> $-\sqrt{B^2 - B + 0,25} - 1$ |
| 3 | $1/(4B\tau + 1)$ | $Bz^2 + 0,5z - B - 0,5$ |
| 4 | $\dfrac{1}{\alpha} arcsh\left[\dfrac{0,5}{B_1 e^{\alpha\tau} - B_2 e^{-\alpha\tau}}\right]$ | $B_1(e^{\alpha z} - e^{-\alpha}) + B_2(e^{-\alpha z} - e^{\alpha}) - 1,$ <br> $B_1 = B_2 + 1/(e^{\alpha} - e^{-\alpha}),\quad \alpha > 0$ |
| 5 | $\sqrt{(\tau+B)^2(\alpha^2-1) + 1 + 2\alpha B + B^2} - \alpha(\tau + B)$ | $\dfrac{Ln\left[(z+B)^2 + 1 + 2\alpha B + B^2\right]}{Ln\left[(1+\alpha)/(1-\alpha)\right]} -$ <br> $\dfrac{Ln\left[(B-1)^2 + 1 + 2\alpha B + B^2\right]}{Ln\left[(1+\alpha)/(1-\alpha)\right]} - 1$ |
| 6 | $\dfrac{1}{\alpha}\left[-d + \sqrt{1 + d^2 + (\alpha\tau + B)^2}\right],$ <br> $d = \dfrac{1 + B^2 - \alpha^2}{2\alpha}$ | $\dfrac{arctg(\alpha z + B)}{arcctg\left[(1 + B^2 - \alpha^2)/(2\alpha)\right]} -$ <br> $\dfrac{arctg(B - \alpha)}{arcctg\left[(1 + B^2 - \alpha^2)/(2\alpha)\right]} - 1$ |
| 7 | $\dfrac{1}{\alpha}\left(\ln\dfrac{1 + \sqrt{1 + 4A^2 e^{2\alpha\tau}}}{2A}\right) - \tau$ | $Ae^{\alpha z} + B,\ \alpha = \ln\dfrac{1 + \sqrt{1 + 4A^2}}{2A}$ |

2. The following class B is determined by the fact that the boundaries move according to the same law:

$$\ell_1(\tau)=\ell(\tau);\ \ell_2(\tau)=1+\ell(\tau);\ \ell(0)=0.$$

Since the movement of the boundaries is interconnected, there is also an interconnection between the functions φ(z) "and " ψ(z). It is expressed by the functional equation:

$$\varphi(\overline{\varphi}(\psi(z))+1)-\psi(z-1)=1.$$

System (7) in this case can only be satisfied by functions that are solutions of equation (10). Here are two previously unknown solutions of class B:

1) $\ell=v\tau;\ \varphi(z)=(1-v)z/2+(1+v)/2-1;$

$$\psi(z)=(1+v)z/2+(1+v)/2-1;$$

2) $\ell(\tau)=\dfrac{1}{\alpha}\ln[(Be^{-\alpha\tau}-Ce^{\alpha\tau})/(B-C)];$

$$\varphi(z)=B(e^{-\alpha z}-1)-C(e^{-\alpha}-1)-1;\ B=C+1/(e^{-\alpha}-1);$$

$$\psi(z)=C(e^{\alpha z}-1)-C(e^{-\alpha}-1)-1.$$

3. For class C solutions, the boundaries move symmetrically in different directions, i.e

$$\ell_1(\tau)=-\ell(\tau);\ \ell_2(\tau)=\ell(\tau).$$

The equation for the relationship of functions φ(z) and ψ(z) here has the form:

$$\varphi(z)=\psi(z)+0{,}5$$

Class C solutions are obtained from class A solutions using the following formulas:

$$\ell(\tau)=\ell_A(\tau);\ \psi(z)=\frac{1}{2}\psi_A(z);\ \varphi(z)=\psi(z)+0{,}5,$$

where the index A denotes the corresponding functions of solutions of the class A.

4. A solution of class D is obtained for the case when both boundaries move uniformly:

$$\ell_1(\tau)=(B_2-B_1)\tau/(B_2+B_1);\ \ell_2(\tau)=(B_2e^{1/c}-B_1)\tau/(B_1+B_2e^{1/c})+1;$$

$$\varphi(z)=C\,Ln(B_1z+D)-C\,Ln(D-B_2)-1;$$

$$\psi(z)=C\,Ln(B_1z+D)-C\,Ln(D-B_2)-1;$$

$$D=(B_1+B_2e^{1/c})/(e^{1/c}-1).$$

The solution number one in Table 1 can be used to study the vibrations of the ropes of load-lifting installations with a uniform ascent (descent) [1, 4, 11–15, 21]. The given solutions of class B can be used in the study of oscillations of flexible gear links [16, 19, 20]. The rest of the solutions are model.

The class of inverse solutions is limited, for example, no solution was obtained for the uniformly accelerated motion of the boundary $l(\tau)=1+v\tau^2$. Obtaining the indicated solution is relevant when describing the longitudinal and transverse vibrations of the ropes of load-lifting installations at the acceleration stage [1].

### 1.3. Method Solution of the Direct Problem

The solution of the direct problem, as a rule, faces great difficulties, because well-known methods for solving functional equations, although sometimes they can be found φ "and " ψ from known ones $l_i(\tau)$, but in a limited range of argument values and in a form that is not very suitable for analytical research.

In this regard, we consider an approximate solution of the functional equation

$$\varphi(\tau+l(\tau))-\varphi(\tau-l(\tau))=1.$$

For an approximate solution of equation (11), it is proposed to use the asymptotic method [17].

For fixed boundaries $l(\tau)=l$ , the solution to (11) is the linear function

$$\varphi_s(z)=\frac{1}{2\ell}z+const.$$

In the case of a slow motion of the boundary l(τ), the "phase" of the wave φ(z) during its run through the system changes insignificantly with respect to $\varphi_s(z)$. It is assumed that φ(z) it has derivatives of any order, and writing $\varphi(\tau+l(\tau))$ in the form of power series in $l(\tau)$ , after substituting them into (1), we obtain a differential equation for slowly changing current "phase" φ(τ)

$$\sum_{k=0}^{\infty}\frac{l^{k+1}}{(k+1)!}\cdot\frac{d^{k+1}\varphi}{d\tau^{k+1}}=1.$$

Since φ(τ) it deviates little from the linear law $\varphi_s(z=\tau)$ during the wave travel time, each next term on the left side of equation (12) is much smaller than the previous one, and its solution must be sought in the form of a series

$$\varphi(\tau)=\sum_{n=0}^{\infty}\varphi_n(\tau).$$

Substituting (13) into (12) and equating the terms of the same order of smallness individually to zero, we obtain for the zeroth approximation

$$\varphi_0(\tau)=\frac{1}{2}\int_0^{\tau}\frac{\partial t}{\ell(t)}.$$

In the case of a linear law of motion of the boundary $l(t)=1+v\tau$, the phase of dynamic natural oscillations is equal to

$$\varphi(z)=\frac{\ln[(vz+1)/(1+v)]}{2v}.$$

Values (14) were compared with the values obtained using the exact solution (Table 1):

$$\varphi(z)=\frac{Ln\left[(vz+1)/(1-v)\right]}{Ln\left[(1+v)/(1-v)\right]}-1$$

The values of the maximum absolute errors Δ of the asymptotic method, depending on the speed of the boundary movement , are given in Table 2.

**Table 2: Error of the asymptotic method depending on the velocity of the boundary**

| v | 0,1 | 0,2 | 0,3 | 0,4 | 0,5 | 0,6 | 0,7 | 0,8 | 0,9 |
|---|---|---|---|---|---|---|---|---|---|
| Δ | 0,002 | 0,006 | 0,013 | 0,023 | 0,036 | 0,053 | 0,073 | 0,100 | 0,139 |

In the interval $v\in[0,1\ ;0,6]$ of error of the approximate method are small. The increase in the error when approaching unity is explained by the fact that the function (15) becomes infinitely large when $v\to1$. Insignificant errors make it possible to apply the described method to solve functional equation (11) in cases where its exact solution is not known.

## 2. Results and Discussion

Using the analytical method of change of variables, the original boundary value problem is reduced to a system of functional-difference equations. The solution of the original problem depends on whether it is possible to solve the given system (7). Vesnitsky A.I. proposed to solve it by the reverse method, i.e., to set functions φ "and " ψ from the resulting system of equations to find the laws of motion of the boundaries. The paper presents five new inverse solutions of the system.

## 3. Conclusion

An approximate asymptotic method for solving the functional equations of system (7) is considered. Under conditions of slow motion of the boundaries, minor errors make it possible to apply this method in cases where the exact solution of the system of functional equations is not known. A technique has been developed that makes it possible to establish the possibility of the occurrence of a steady state resonance phenomenon and the phenomenon of passage through a resonance, as well as to calculate the amplitude of the oscillations arising in this case.

### 3.1. Current & Future Developments

An analytical method for solving the wave equation (the method of changing variables in a system of functional-difference equations) has been developed, which makes it possible to obtain a solution with a wider class of conditions on moving boundaries, different from the boundary conditions of the first kind. New mathematical models for describing oscillatory processes in systems with moving boundaries are proposed and investigated. Research has been brought to graphs and tables, which makes it possible to use the results in automated systems for designing mechanisms containing oscillating elements with moving boundaries.

In the Matlab environment, the TB-ANALYSIS software package has been developed, which implements numerical-analytical and approximately-analytical methods, designed to solve a certain class of boundary value problems with moving boundaries and study the resonant properties of objects whose state is described by these boundary value problems. The testing of the software package was carried out, which showed its effectiveness in performing the specified tasks. A certificate of registration of an electronic resource - a patent for the developed software package [29].

The above solutions can be used in the study of vibrations of ropes of lifting installations with a uniform rise (descent), flexible links of transmission (for example, a belt drive), etc.

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