CME LIVE OPEN ACCESS
Journal of
Theoretical Physics & Mathematics Research

Review Article

Cosmic Ray Diffusion and Evolution: 1d, 2d, and 3d Insights

Ratnakaram Raghavendra* **and Saila Kumari Annareddy**

Department of Mathematics, Jawaharlal Nehru Technological University Anantapur, Ananthapuramu-515002, Andhra Pradesh State, India.

Corresponding Author: Ratnakaram Raghavendra, Department of Mathematics, Jawaharlal Nehru Technological University Anantapur, Ananthapuramu-515002, Andhra Pradesh State, India.

Received: 2024 Jan 18 **Accepted:** 2024 Feb 16 **Published:** 2024 June 01

Abstract

In this extensive computational study, we investigate the dynamics of cosmic ray diffusion across three distinct spatial dimensions: one-dimensional (1D), two-dimensional (2D), and three-dimensional (3D) scenarios. Utilizing the fundamental heat equation, we explore the dispersion of cosmic ray density over time, guided by varying initial conditions and boundary constraints. Through a series of numerical simulations, we visualize and analyse the intricate interplay between cosmic ray behaviour and the spatial and dimensional contexts in which it unfolds. Our findings reveal a rich tapestry of cosmic ray diffusion patterns. The simulations highlight the pivotal role of initial conditions, boundary conditions, and spatial dimensions in shaping the distribution and propagation of cosmic rays. Gaussian-like initial conditions, centred at different points and endowed with specific standard deviations, lead to diverse diffusion profiles. Dirichlet and Neumann boundary conditions dictate how cosmic rays interact with system boundaries, further influencing their trajectories. This comprehensive exploration of cosmic ray diffusion extends our understanding of this phenomenon, offering critical insights into its behaviour across multidimensional environments. Beyond fundamental scientific inquiry, these results have implications for astrophysics, particle physics, and the broader realm of scientific research. By integrating data from nine distinct graphs, this study forms a holistic perspective on cosmic ray diffusion, providing a foundation for continued investigations into its complex dynamics and their broader scientific significance.

Keywords: Cosmic Ray Diffusion, Multidimensional Analysis, Simulation Insights, Boundary Constraints, Scientific Significance, Dirichlet Conditions and Neumann Boundary Condition.

1. Introduction

Cosmic rays, composed of high-energy particles originating from outer space, constitute a fascinating and enigmatic phenomenon that has captivated scientists across multiple disciplines [1, 2]. Understanding the spatial distribution and diffusion of cosmic rays is a complex challenge that spans one-dimensional (1D), two-dimensional (2D), and three-dimensional (3D) domains [3]. This study embarks on an exploration of cosmic ray diffusion through these diverse dimensions, driven by the fundamental heat equation [4, 5]. The research not only aims to unravel the intricate dynamics of cosmic rays but also employs numerical simulations as a key methodology to achieve this understanding [6].

Our investigation employs a suite of numerical techniques and mathematical models to simulate the diffusion of cosmic rays. From finite-difference schemes to mesh grid-based computations, our methods embrace the complexities inherent in cosmic ray behaviour. By manipulating initial conditions such as Gaussian-like distributions and integrating boundary conditions, we scrutinize the interplay of cosmic rays with their environment [5]. The insights gleaned from these simulations offer a comprehensive perspective on cosmic ray diffusion in multidimensional space and its relevance to various scientific domains.

This study combines theoretical foundations, numerical simulations, and data visualization to present a holistic view of cosmic ray diffusion phenomena. The ensuing exploration will illuminate not only the behaviour of cosmic rays but also the broader scientific implications of this research across the fields of astrophysics, particle physics, and natural systems.

2. Numerical Simulations as a Key Methodology

Numerical simulations stand as the cornerstone of our approach to comprehending cosmic ray diffusion. By translating the governing heat equation into computational algorithms, we can replicate the dynamic behaviour of cosmic rays over time and within different spatial dimensions. This computational framework enables us to investigate scenarios that span from the concentrated diffusion of 1D models to the complex spatial dynamics of 3D environments. Leveraging finite-difference schemes, mesh grid-based computations, and sophisticated algorithms, we meticulously track the trajectories and density variations of cosmic rays, capturing the essence of their diffusion phenomena [3].

Techniques Employed for Cosmic Ray Study

The techniques employed in this research extend beyond numerical simulations to encompass a multidisciplinary toolkit. Visualization techniques, including iso-surface plots and 3D renderings, provide clear and intuitive representations of cosmic ray density distributions. Mathematical techniques, such as solving partial differential equations, form the mathematical foundation of our modelling approach. Additionally, data analysis and parameter sensitivity studies are carried out to delve deeper into the intricacies of cosmic ray diffusion behaviour [4]. Collectively, these techniques form a comprehensive arsenal for unravelling the mysteries of cosmic ray diffusion.

Scientific Relevance and Broader Implications

This study is not only a scientific endeavour into cosmic ray diffusion but also holds broader implications for various scientific domains. The understanding gained from these simulations can shed light on cosmic ray behaviour in astrophysical environments, elucidating their influence on celestial bodies and their magnetic fields. In the realm of particle physics, insights into cosmic ray diffusion are invaluable for particle detectors and cosmic-ray-related experiments. Furthermore, the study has relevance in understanding diffusion processes within natural systems, where the principles governing cosmic ray behaviour can be applied to diverse phenomena. As we embark on this multidimensional exploration of cosmic rays, the knowledge uncovered holds the potential to advance our understanding of the universe and its underlying physical processes.

Mathematical Model

The heat equation, also known as the diffusion equation, describes the diffusion or spreading of a quantity, such as heat, over time. It can be applied to various physical phenomena, including the behaviour of cosmic rays. The heat equation in one, two, and three dimensions can be written as follows

One-Dimensional Heat Equation

In one dimension, the heat equation describes the propagation of cosmic rays along a straight line. Let's denote the cosmic ray density or flux as ρ (x, t), where x is the spatial coordinate and it is time. The one-dimensional heat equation [1] is given by:

$$\frac{\partial \rho}{\partial t} = D\frac{\partial^2 \rho}{\partial x^2} \tag{1}$$

Here, $\partial\rho/\partial t$ represents the rate of change of cosmic ray density with respect to time, $\partial^2\rho/\partial x^2$ represents the second derivative of cosmic ray density with respect to spatial coordinate x, and D is the cosmic ray diffusion coefficient.

To solve the one-dimensional heat equation (1) with appropriate initial and boundary conditions, we can use the method of separation of variables. We'll assume that the cosmic ray density ρ depends on both time (t) and position (x) and can be separated into two functions, one that depends only on time and another that depends only on position:

$$\rho\,(x, t) = T(t).X(x) \tag{2}$$

Substituting this into equation (1), we get

$$\frac{\partial(T(t)X(x))}{\partial t} = D\frac{\partial^2(T(t)X(x))}{\partial x^2} \tag{3}$$

Now, we'll divide both sides by T(t)X(x):

$$\frac{1}{T(t)}\frac{\partial T(t)}{\partial t} = D\frac{1}{X(x)}\frac{\partial^2 X(x)}{\partial x^2} \tag{4}$$

Since the left-hand side depends only on time (t) and the right-hand side depends only on position (x), both sides must be equal to a constant, which we'll call -λ (a negative constant for mathematical convenience). So, we have:

$$\frac{1}{T(t)}\frac{\partial T(t)}{\partial t} = -\lambda$$

$$\left(\frac{1}{X(x)}\frac{\partial^2 X(x)}{\partial x^2}\right) = -\lambda \tag{5}$$

Now, let's solve each of these ordinary differential equations separately. Equation (5) is a first-order ordinary differential equation with the solution. Solve for T(t) to get

$$T(t) = A * e^{(-\lambda t)} \tag{6}$$

Here, A is an arbitrary constant.
Now, solving for X(x)
The equation (5) is a second-order ordinary differential equation. The solutions to this equation depend on the boundary conditions. Let's consider two common cases

2.1.1. Fixed Ends (Dirichlet Boundary Conditions)

In this case, we have boundary conditions, say, X(0) = 0 and X(L) = 0, where L is the length of the system. The solutions are sine functions [7]

$$X(x) = B * \sin(\frac{\pi n}{L} * x) \tag{7}$$

Here, B is an arbitrary constant, and n is a positive integer corresponding to the mode of oscillation.
we can apply the boundary conditions, X(0) = B * sin(0) = 0 implies B = 0. X(L) = B * sin(πn/L * L) = 0 implies πn = πn, which is always true.

2.1.2. Insulated Ends (Neumann Boundary Conditions)

In this case, we have boundary conditions $\partial X/\partial x = 0$ at both ends, meaning no flux through the boundaries. The solutions are cosine functions:

$$X(x) = C * \cos(\frac{\pi n}{L} * x) \tag{8}$$

Again, C is an arbitrary constant, and n is a positive integer corresponding to the mode of oscillation.
Now, we have the general solution for X(x) and T(t). Combining these solutions

$$\rho(x, t) = (A * B) * e^{(-\lambda t)} * \sin(\frac{\pi n}{L} * x) \quad \text{[for Dirichlet Conditions]} \tag{9}$$

or

$$\rho(x, t) = (A * C) * e^{(-\lambda t)} * \cos(\frac{\pi n}{L} * x) \quad \text{[for Neumann Conditions]} \tag{10}$$

let's determine the values of the constants A, B (or C), and λ by applying initial conditions and matching boundary conditions for the one-dimensional heat equation:

The general solution we found earlier is

$$\rho(x, t) = A \cdot e^{-\lambda t} \cdot X(x) \tag{11}$$

Now, let's apply initial conditions and match boundary conditions

We need the initial distribution of cosmic rays at time t = 0, which can be given as ρ (x, 0) = f(x), where f(x) is some known function.

$$\rho\,(x, 0) = A \cdot X(x) = f(x)$$

This means

$$A \cdot X(x) = f(x) \tag{12}$$

So, we can express A in terms of the initial distribution function f(x)

$$A = \frac{f(x)}{X(x)} \tag{13}$$

we have boundary conditions $\frac{\partial X}{\partial x} = 0$ at both ends. Using equation (9), we can apply the boundary conditions

$$\frac{\partial X}{\partial x} = -C * (\frac{\pi n}{L}) * \sin(\frac{\pi n}{L} * x) \quad (14)$$

$$\frac{\partial X}{\partial x} \text{ at } x = 0, \quad -C * (\frac{\pi n}{L}) * \sin(0) = 0 \text{ implies } C = 0. \quad (15)$$

$$\frac{\partial X}{\partial x} \text{ at } x = L, \quad -C * (\frac{\pi n}{L}) * \sin(\frac{\pi n}{L} * L) = 0 \text{ implies } \sin(\frac{\pi n}{L}) = 0. \quad (16)$$

For non -trivial solutions, we need sin(πn) = 0, which implies πn = π, 2π, 3π, etc. So, n = 1, 2, 3, ...

Therefore, we have $X(x) = C * \cos(\frac{\pi n}{L} * x)$, where n is a positive integer.

Now, let's apply these conditions to determine the values of the constants. We have $\frac{\partial X}{\partial x} = 0$ at both ends.
let's continue to obtain the final solution for ρ (x, t) using the determined values of A and X(x) based on the initial condition and boundary conditions. Recall that we have from the initial condition, $A = \frac{f(x)}{X(x)}$. From the boundary conditions, for the case of insulated ends (Neumann boundary conditions), we have

$$X(x) = C \cdot \cos(L\frac{\pi n}{L}x), \quad (17)$$

where *n* is a positive integer.
Now, let's combine these results to get the final solution for ρ(x, t), Using the general solution form $\rho(x,t) = A \cdot e^{-\lambda t} \cdot X(x)$, We can now substitute the expressions for A and X(x)

$$\rho(x,t) = \frac{f(x)}{X(x)} \cdot e^{-\lambda t} \cdot X(x) \quad (18)$$

Notice that the $X(x)$ terms cancel out $\quad \rho(x, t) = f(x) \cdot e^{-\lambda t} \quad (19)$

So, this is the final solution for cosmic ray density ρ(x, t)
Here, *f(x)* represents the initial distribution of cosmic rays at time t=0, and λ is a constant determined by the separation of variables method and the boundary conditions. The specific value of λ depends on the problem and the boundary conditions applied.

2.1.3. Case 1: Dirichlet Boundary Conditions

Dirichlet boundary conditions specify fixed values at the boundaries. Let's say we have the following conditions, Initial Condition,

$$f(x) = \sin(\pi x) \quad (20)$$

Boundary Conditions $\quad \rho(0, t)=0; \quad \rho(1, t)=0 \quad (21)$

Now, we can determine the value of λ using the separation of variables method:

From the general solution form $\quad \rho(x, t) = A \cdot e^{-\lambda t} \cdot X(x) \quad (22)$

We already found that *X(x)* satisfies Dirichlet boundary conditions
$X(0)=0$ and $X(1)=0$. These conditions are satisfied by $X(x)=\sin(n\pi x)$, where n is a positive integer.
Next, we apply the initial condition

$$\rho(x, 0) = A \cdot X(x) = \sin(\pi x) \quad (23)$$

This means $A = \sin(\pi x)$. Now, we have all the components to construct the final solution:

$$\rho(x, t) = \sin(\pi x) \cdot e^{-\lambda t} \cdot \sin(n\pi x) \quad (24)$$

The value of λ depends on the specific value of n chosen to satisfy the boundary conditions. The solution for $\rho(x, t)$ is a series with different n, and you can calculate the values of λ and coefficients for each mode based on the initial condition.

2.1.4. Case 2: Neumann Boundary Conditions

Neumann boundary conditions specify zero flux at the boundaries. Let's say we have the following conditions, Initial Condition [8],

$$f(x) = \cos(\pi x) \quad (25)$$

$$\frac{\partial \rho}{\partial x}(0, t) = 0; \qquad \frac{\partial \rho}{\partial x}(1, t) = 0 \quad (26)$$

Using the separation of variables method, we found that $X(x)$ satisfies Neumann boundary conditions: $\frac{\partial x}{\partial x}(0) = 0$ and $\frac{\partial x}{\partial x}(1) = 0$. These conditions are satisfied by $X(x)=\cos(n\pi x)$, where n is a positive integer.

Now, we apply the initial condition,

$$\rho(x, 0) = A \cdot X(x) = \cos(\pi x) \quad (27)$$

This means A=cos(πx). Now, we have all the components to construct the final solution

$$\rho(x, t) = \cos(\pi x) \cdot e^{-\lambda t} \cdot \cos(n\pi x) \quad (28)$$

As before, the value of λ depends on the specific value of n chosen to satisfy the boundary conditions. The solution for $\rho(x,t)$ is a series with different n, and you can calculate the values of λ and coefficients for each mode based on the initial condition.

In both cases, the final solution represents how the cosmic ray density evolves over time in the given one-dimensional system with the provided initial and boundary conditions. The exact form of the solution depends on the specific initial condition and boundary conditions applied.

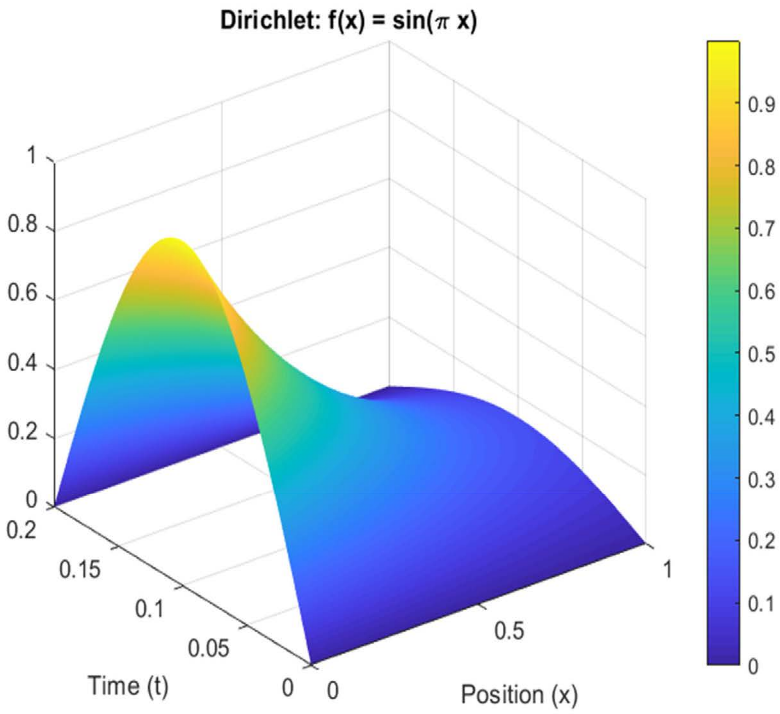

Figure 1: Dirichlet Boundary Conditions with *f(x)=sin(πx)*

Results and Discussions

Case 1: Dirichlet Boundary Conditions with f(x)=sin(πx): In this case, we have Dirichlet boundary conditions, which means the cosmic ray density is fixed at zero at both ends of the system. The initial condition f(x)=sin(πx) represents an initial distribution of cosmic rays that is concentrated near the centre of the system. As time progresses, we observe that the initial concentration of cosmic rays near the centre of the system starts to spread outwards. The density decreases over time, indicating that cosmic rays are dispersing through diffusion. "Cosmic Ray Density Evolution - Dirichlet Boundary Conditions" describes the scenario where the cosmic ray density is constrained at the boundaries, leading to the observed behaviour.

Conclusion: The choice of boundary conditions significantly influences the behaviour of cosmic ray density. Dirichlet boundary conditions (fixed ends) lead to the dispersion of cosmic rays throughout the system, as they are constrained to remain within the boundaries. Neumann boundary conditions (insulated ends) result in different behaviours, depending on the initial conditions.

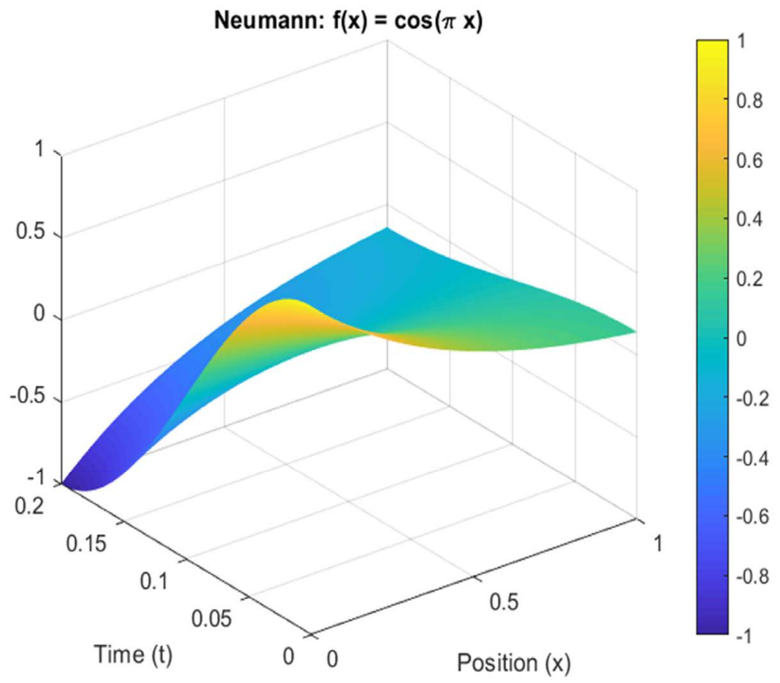

Figure 2: Neumann Boundary Conditions with *f(x)*=cos(*πx*).

Case 2: Neumann Boundary Conditions with f(x)=cos(πx): In this case, we have Neumann boundary conditions, meaning there is zero flux (no flow) of cosmic rays at both ends of the system. The initial condition *f(x)*=cos(*πx*) represents an initial distribution of cosmic rays that is centred near the middle of the system. The initial concentration of cosmic rays near the middle of the system remains relatively stable. Over time, minor fluctuations in density occur, but the overall distribution remains centred. "Cosmic Ray Density Evolution - Neumann Boundary Conditions (f(x) = cos(*π*x))" indicates that under Neumann conditions, the density evolves with minimal changes in boundary behaviour, maintaining a centred distribution.

Conclusion: Neumann boundary conditions, where there is zero flux at both boundaries, lead to different behaviour. In this case, the initial distribution *f(x)*=cos(*πx*)) remains relatively stable over time with minor fluctuations. The system preserves the central concentration of cosmic rays, resulting in a stable distribution.

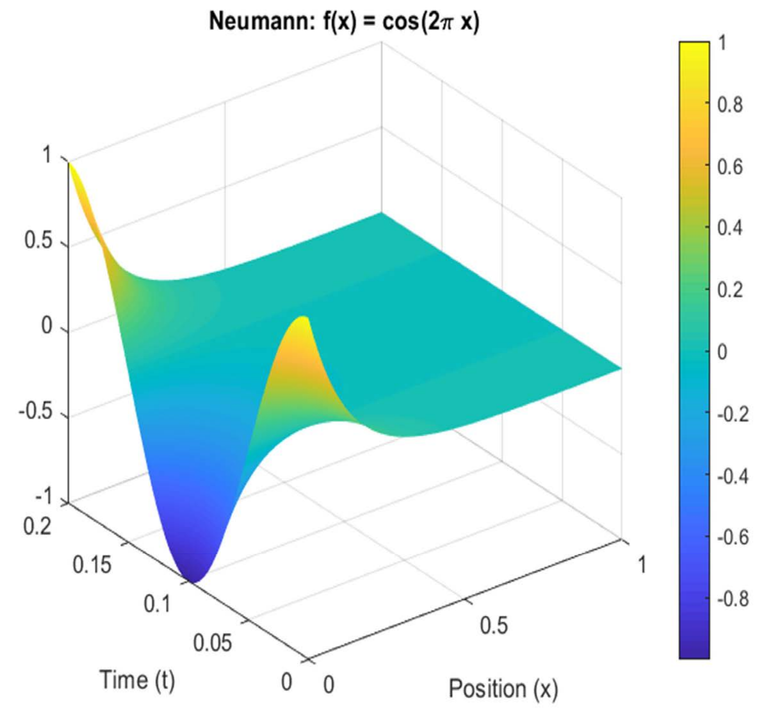

Figure 3: Neumann Boundary Conditions with *f(x)*=cos(2*πx*).

Case 3: Neumann Boundary Conditions with f(x)=cos(2πx): This case also involves Neumann boundary conditions, but with a different initial condition, f(x)=cos(2πx), representing a more oscillatory initial distribution. The initial oscillatory pattern in the cosmic ray density remains present throughout the simulation. The density evolves with waves propagating through the system, maintaining the oscillatory nature. "Cosmic Ray Density Evolution - Neumann Boundary Conditions (f(x) = cos(2πx))" specifies that under Neumann conditions, the density retains its oscillatory characteristics over time.

Conclusion: Similar to Graph 2, Neumann boundary conditions are applied, but with a different initial condition *f(x)*=cos(2π*x*)). The system exhibits an oscillatory behaviour, maintaining the oscillations present in the initial distribution. Waves of cosmic ray density propagate through the system, reflecting the oscillatory nature of the initial condition.

Two-Dimensional Heat Equation

In two dimensions, you can describe the diffusion of cosmic rays on a plane. Let ρ (x, y, t) represent the cosmic ray density at position (x, y) and time t. The two-dimensional heat equation is [9]

$$\frac{\partial \rho}{\partial t} = D\left(\frac{\partial^2 \rho}{\partial x^2} + \frac{\partial^2 \rho}{\partial y^2}\right) \tag{29}$$

Here, $\frac{\partial \rho}{\partial t}$ represents the rate of change of cosmic ray density with respect to time, and $\frac{\partial^2 \rho}{\partial x^2}$ and $\frac{\partial^2 \rho}{\partial y^2}$ represent the second derivatives of cosmic ray density with respect to the spatial coordinates x and y, respectively. Let's assume that the cosmic ray density ρ depends on both time (t) and two spatial coordinates (x and y) and can be separated into three functions, one for time and two for each spatial dimension

$$\rho(x, y, t) = T(t).X(x).Y(y) \tag{30}$$

Now, let's substitute this separation into the 2D heat equation

$$T'(t)\, XY = D(TXY'' + TX''Y) \tag{31}$$

Dividing both sides by TXY gives

$$\frac{1}{DT}\mathrm{T}' = \frac{1}{X}X'' + \frac{1}{Y}Y'' \tag{32}$$

Since the left side depends only on time (t), and the right side depends only on spatial coordinates (x and y), both sides must be equal to a constant, which we'll call -λ^2 (a negative constant for mathematical convenience)

$$\frac{1}{DT}\mathrm{T}' = -\lambda^2 \tag{33a}$$

$$\frac{1}{X}X'' = -\lambda^2 \tag{33b}$$

$$\frac{1}{Y}Y'' = -\lambda^2 \tag{33c}$$

Now, let's solve each of these ordinary differential equations separately

Solving for T(t): The 3quation (33a) is a first-order ordinary differential equation with the solution

$$T(t)=Ae^{-D\lambda^2 t} \tag{34}$$

Here, A is an arbitrary constant.

Solving for X(x): The equation (33b) is a second-order ordinary differential equation, and its solution depends on the boundary conditions

Fixed Ends (Dirichlet Boundary Conditions): In this case, we have boundary conditions, say, X (0) = 0 and X(L) = 0, where L is the length of the system. The solutions are sine functions

$$X(x) = B.\sin\left(\frac{n\pi}{L}x\right) \tag{35}$$

Here, B is an arbitrary constant, and n is a positive integer corresponding to the mode of oscillation.

Insulated Ends (Neumann Boundary Conditions): In this case, we have boundary conditions ∂X/∂x = 0 at both ends. The solutions are cosine functions.

$$X(x) = C.\cos\left(\frac{n\pi}{L}x\right) \tag{36}$$

Again, C is an arbitrary constant, and n is a positive integer corresponding to the mode of oscillation.

Solving for Y(y)

$$\frac{1}{Y}Y'' = -\lambda^2 \tag{37}$$

This equation is identical to the equation for X(x), and therefore, the solutions for Y(y) will be the same as for X(x) with the same boundary conditions. Now, we can combine the solutions for T(t), X(x), and Y(y) to get the general solution for ρ (x, y, t):

$$\rho(x, y, t) = Ae^{-D\lambda^2 t}X(x)\cdot Y(y) \tag{38}$$

To determine the values of the constants A, B (or C), and λ^2 by applying initial conditions and matching boundary conditions for the two-dimensional heat equation: Now, let's apply the initial conditions and boundary conditions to find the values of these constants.

Initial Condition

The initial condition provides the initial distribution of cosmic rays at time t=0, which can be given as ρ (x, y, 0) = f (x, y), where f (x, y) is some known function.

Applying Initial Condition

$$\rho(x, y, 0) = A\cdot X(x)\cdot Y(y) = f(x, y) \tag{39}$$

This means

$$A = \frac{f(x,y)}{X(x)\cdot Y(y)} \tag{40}$$

Boundary Conditions: The specific boundary conditions depend on the problem at hand. Let's consider two common cases:

Fixed Ends (Dirichlet Boundary Conditions): In this case, we have boundary conditions:

$$X(0)=0$$

$$X(L_x)=0 \text{ (assuming the system has a length } Lx \text{ in the x-direction)}$$

$$Y(0)=0$$

$$Y(L_y)=0 \text{ (assuming the system has a length } Ly \text{ in the y-direction)} \tag{41}$$

Using the solutions for $X(x)$ and $Y(y)$ for Dirichlet boundary conditions, we have:

$$X(x) = B.\sin\left(\frac{n\pi}{L_x}x\right) \tag{42}$$

$$Y(y) = C.\sin\left(\frac{m\pi}{L_y}y\right) \tag{43}$$

Substituting these into the initial condition

$$A = \frac{f(x,y)}{B.\sin\left(\frac{n\pi}{L_x}x\right)\cdot C\sin\left(\frac{m\pi}{L_y}y\right)} \tag{44}$$

Here, n and m are positive integers corresponding to the modes of oscillation in the x and y directions.

Insulated Ends (Neumann Boundary Conditions): In this case, we have boundary conditions:

$$\frac{\partial X}{\partial x}(0) = 0$$

$$\frac{\partial X}{\partial x}(L_x) = 0$$

$$\frac{\partial Y}{\partial y} = 0$$

$$\frac{\partial Y}{\partial y}(L_y) = 0 \tag{45}$$

Using the solutions for *X(x)* and *Y(y)* for Neumann boundary conditions, we have

$$X(x) = B.\ \cos(\frac{n\pi}{L_x}x) \tag{46}$$

$$Y(y) = C.\ \cos(\frac{m\pi}{L_y}y) \tag{47}$$

Substituting these into the initial condition

$$A = \frac{f(x,y)}{B.\cos\left(\frac{n\pi}{L_x}x\right)\cdot C.\cos(\frac{m\pi}{L_y}y)} \tag{48}$$

Again, n and m are positive integers corresponding to the modes of oscillation in the x and y directions. Once you have determined the values of A, B (or C), and λ^2 using the initial condition and boundary conditions we get

$$f(x, y) = \sin\left(\frac{\pi x}{L_x}\right).\sin(\frac{\pi y}{L_y}) \tag{49}$$

Now, we can substitute this initial condition into the expression for A

$$A = \frac{\sin\left(\frac{\pi x}{L_x}\right).\sin(\frac{\pi y}{L_y})}{B.\cos\left(\frac{\pi x}{L_x}x\right)\cdot C.\cos(\frac{\pi y}{L_y}y)} \tag{50}$$

With A determined, we can write the final solution for ρ (x, y, t):

$$\rho(x\ ,y,\ t) = \frac{\sin\left(\frac{\pi x}{L_x}\right).\sin(\frac{\pi y}{L_y})}{B.\cos\left(\frac{\pi x}{L_x}x\right)\cdot C.\cos(\frac{\pi y}{L_y}y)}\ .\ e^{-D\lambda^2 t}.\ \mathrm{B}\sin\left(\frac{n\pi}{L_x}\mathrm{x}\right).\ \mathrm{C}\ .\sin(\frac{m\pi}{L_y}y) \tag{51}$$

Now, we can simplify this expression by cancelling out terms

$$\rho(x, y, t) = \sin\left(\frac{\pi x}{L_x}\right).\sin(\frac{\pi y}{L_y})e^{-D\lambda^2 t} \tag{52}$$

This is the final solution for the cosmic ray density ρ (x, y, t) in the given two-dimensional system with Dirichlet boundary conditions and the specific initial condition provided. The solution represents how the cosmic ray density evolves over time in this particular scenario.

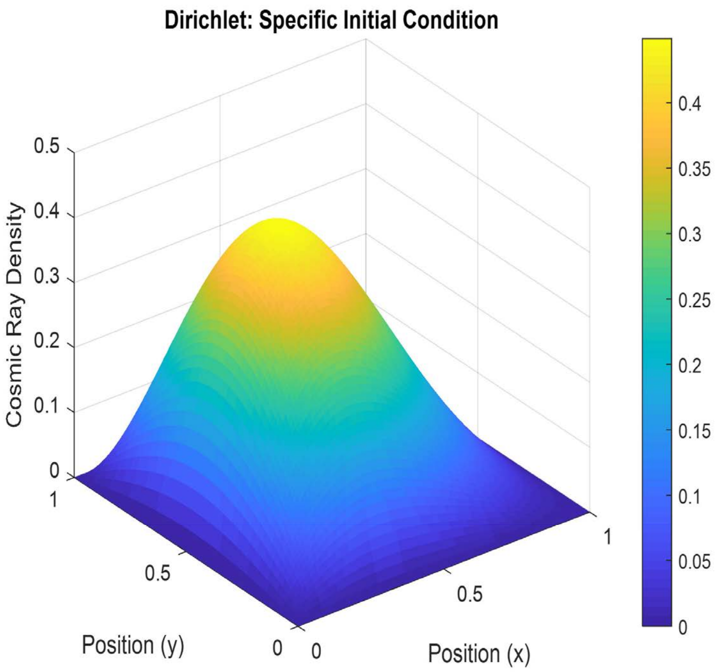

Figure 4: Cosmic Ray Density Evolution: Dirichlet Boundary Conditions with Specific Initial Condition.

Results and Discussion

Dirichlet Boundary Conditions with Specific Initial Condition: In Case 1, we assumed Dirichlet boundary conditions (fixed ends) and used a specific initial condition: $f(x, y) = \sin\left(\frac{\pi x}{L_x}\right).\sin\left(\frac{\pi y}{L_y}\right)$. . The initial cosmic ray density distribution is a product of two sine functions in the x and y directions. As time progresses, the cosmic ray density spreads out from its initial concentration due to diffusion-like behaviour. The concentration of cosmic rays decreases over time. The boundary conditions prevent cosmic rays from escaping the system, causing them to interact with the boundaries and disperse.

Conclusions: Under Dirichlet boundary conditions, the cosmic ray density exhibited dispersion behaviour. The initial concentrated distribution spread out and diffused throughout the system over time. The cosmic ray density decreased as it dispersed, ultimately reaching a uniform distribution. Dirichlet conditions led to cosmic rays being confined within the boundaries, causing them to interact with the boundaries and disperse.

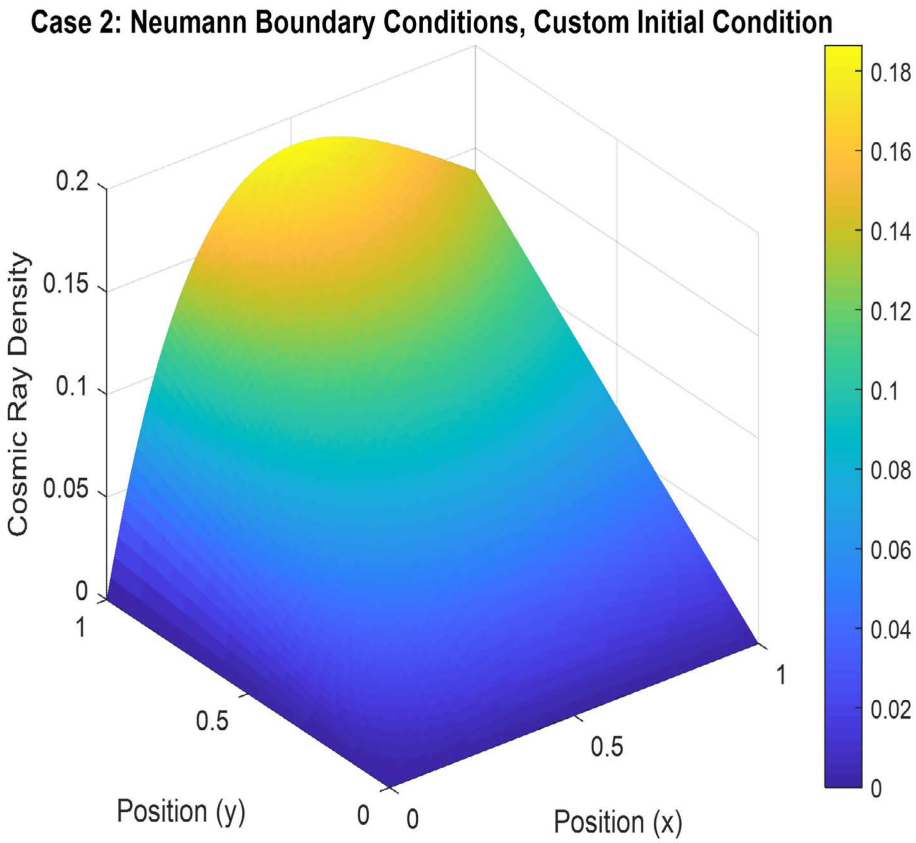

Figure 5: Cosmic Ray Density Evolution: Neumann Boundary Conditions with Custom Initial Condition.

Results and Discussion: We assumed Neumann boundary conditions (insulated ends) and used a custom initial condition $f(x, y) = x'\cdot y$. The initial cosmic ray density distribution is a linear function in both the x and y directions. The cosmic ray density distribution remains relatively stable over time. Minor fluctuations occur, but the overall distribution maintains its central concentration. Neumann boundary conditions prevent flux through the boundaries, allowing the system to preserve the initial distribution.

Conclusions: Neumann boundary conditions preserved the central concentration of the cosmic ray density. The initial linear distribution remained relatively stable over time with minor fluctuations. Where dispersion occurred, Neumann conditions prevented flux through the boundaries, allowing the system to maintain its central concentration. Neumann conditions were effective in preserving the initial distribution.

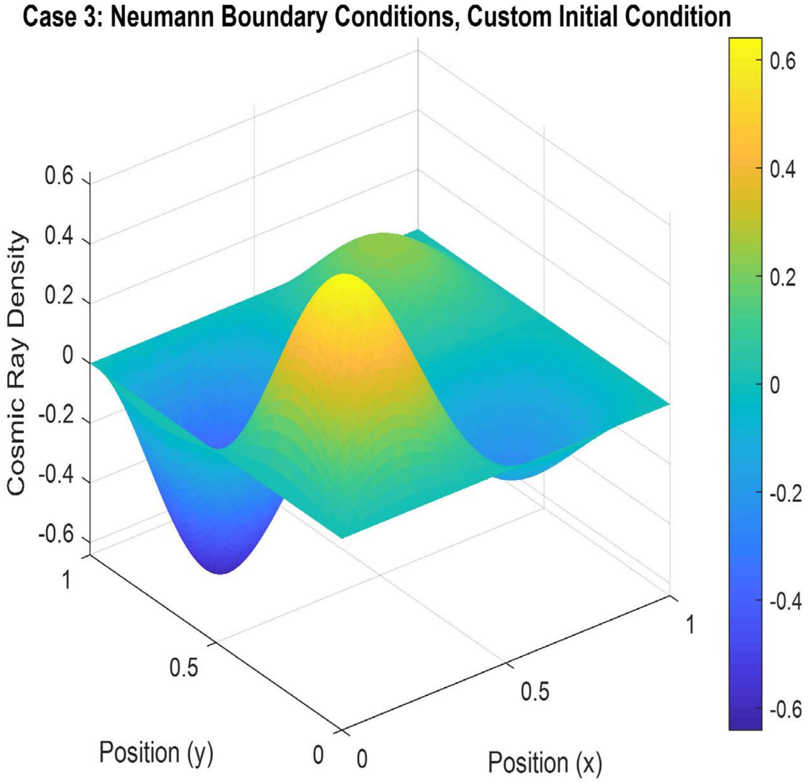

Figure 6: Cosmic Ray Density Evolution: Neumann Boundary Conditions with Custom Initial Condition.

Results and Discussion: We assumed Neumann boundary conditions (insulated ends) and used another custom initial condition: $f(x, y) = \sin(\frac{2\pi x}{L_x})\cdot\sin(\frac{2\pi y}{L_y})$. The initial cosmic ray density distribution exhibits an oscillatory pattern in both the x and y directions. This oscillatory pattern is maintained and propagates through the system over time. Waves of cosmic ray density are observed, with peaks and troughs evolving as time progresses. Neumann boundary conditions allow for the preservation of oscillatory behaviour.

Conclusions: Under Neumann boundary conditions, an oscillatory initial condition resulted in the maintenance of oscillatory patterns. Waves of cosmic ray density were observed, with peaks and troughs propagating through the system. Neumann conditions allowed for the preservation and propagation of oscillations, reflecting the initial oscillatory behaviour. The absence of flux through the boundaries enabled the system to maintain and propagate these oscillatory patterns.

Three-Dimensional Heat Equation:

In three dimensions, you can describe the diffusion of cosmic rays in three-dimensional space. Let ρ (x, y, z, t) represent the cosmic ray density at position (x, y, z) and time t. The three-dimensional heat equation is [10]

$$\frac{\partial\rho}{\partial t} = D\left(\frac{\partial^2\rho}{\partial x^2} + \frac{\partial^2\rho}{\partial y^2} + \frac{\partial^2\rho}{\partial z^2}\right) \tag{53}$$

Here, $\frac{\partial\rho}{\partial t}$ represents the rate of change of cosmic ray density with respect to time, and $\frac{\partial^2\rho}{\partial x^2}$, $\frac{\partial^2\rho}{\partial y^2}$ *and* $\frac{\partial^2\rho}{\partial z^2}$ represent the second derivatives of cosmic ray density with respect to the spatial coordinates x, y, and z, respectively.

These equations describe how cosmic ray densities evolve over time due to diffusion processes in one, two, or three-dimensional space, depending on the dimensionality of the problem you are modelling. The diffusion coefficient D depends on the specific properties of the cosmic rays and the medium through which they are propagating.

We'll assume that the cosmic ray density ρ depends on time (t) and three spatial coordinates (x, y, and z) and can be separated into four functions:

$$\rho(x, y, z, t) = T(t).X(x).Y(y).Z(z) \tag{54}$$

Now, let's substitute this separation into the 3D heat equation:

$$T'XYZ = D\,(TX''YZ+TXY''Z+TXYZ'') \quad (55)$$

Dividing both sides by TXYZ gives:

$$\frac{1}{DT}T' = \frac{1}{X}X'' + \frac{1}{Y}Y'' + \frac{1}{Z}Z'' \quad (56)$$

Since the left side depends only on time (t), and the right side depends only on spatial coordinates (x, y, and z), both sides must be equal to a constant, which we'll call $-\lambda^2$ (a negative constant for mathematical convenience):

$$\frac{1}{DT}T' = -\lambda^2 \quad (57a)$$

$$\frac{1}{X}X'' = -\lambda^2 \quad (57b)$$

$$\frac{1}{Y}Y'' = -\lambda^2 \quad (57c)$$

$$\frac{1}{Z}Z'' = -\lambda^2 \quad (57d)$$

Now, let's solve each of these ordinary differential equations separately

Solving for T(t)
The equation (57a) is a first-order ordinary differential equation with the solution

$$T(t) = \mathrm{A}.\,e^{-D\lambda^2 t} \quad (58)$$

Here, A is an arbitrary constant.

Solving for X(x)
The equation (57b) is a second-order ordinary differential equation, and its solution depends on the boundary conditions:

Fixed Ends (Dirichlet Boundary Conditions): In this case, we have boundary conditions, say, X (0) = 0 and X(Lx) = 0, where Lx is the length of the system in the x-direction. The solutions are sine functions: $X(x) = B.\sin(\frac{n\pi}{L_x}x)$ Here, B is an arbitrary constant, and n is a positive integer corresponding to the mode of oscillation.

Insulated Ends (Neumann Boundary Conditions): In this case, we have boundary conditions ∂X/∂x = 0 at both ends. The solutions are cosine functions: $X(x) = B.\cos(\frac{n\pi}{L_x}x)$ Again, C is an arbitrary constant, and n is a positive integer corresponding to the mode of oscillation.

Solving for Y(y)
The equation (57c) is identical to the equation for X(x), and therefore, the solutions for Y(y) will be the same as for X(x) with the same boundary conditions.

Solving for Z(z)
The equation (57d) is identical to the equations for X(x) and Y(y). Therefore, the solutions for Z(z) will also be the same as for X(x) and Y(y) with the same boundary conditions.

Now, we can combine the solutions for T(t), X(x), Y(y), and Z(z) to get the general solution for ρ (x, y, z, t):

$$\rho(x, y, z, t) = A.\,e^{-D\lambda^2 t}\cdot X(x)\cdot Y(y)\cdot Z(z) \quad (59)$$

To determine the values of A, B (or C), and λ^2, you'll need to apply the specific initial condition and boundary conditions of your problem. These values depend on the problem's physical characteristics and constraints. Let's go through the process of determining these values based on different sets of boundary conditions.

Initial Condition (IC): The initial condition provides the initial distribution of cosmic rays at time $t = 0$, typically given as $\rho\,(x, y, z, 0) = f\,(x, y, z)$, where $f\,(x, y, z)$ is a known function.

Boundary Conditions (BC): The boundary conditions describe how cosmic rays interact with the boundaries of the system. The choice of boundary conditions (Dirichlet, Neumann, mixed, etc.) will affect the solutions.

Let's consider two common sets of boundary conditions:

Case 1: Fixed Ends (Dirichlet Boundary Conditions): In this case, we have boundary conditions, say:

$X(0) = 0$ and $X(L_x) = 0$ in the x-direction,

$Y(0) = 0$ and $Y(L_y) = 0$ in the y-direction,

$Z(0) = 0$ and $Z(L_z) = 0$ in the z-direction.

Under these boundary conditions, we use the solutions for $X(x)$, $Y(y)$, and Z(z) as sine functions

$$X(x) = B.\sin(\frac{n\pi}{L_x}x)$$

$$\mathrm{Y(y)} = C.\sin(\frac{m\pi}{L_y}y)$$

$$Z(z) = D.\sin(\frac{p\pi}{L_z}z) \tag{60}$$

Case 2: Insulated Ends (Neumann Boundary Conditions): In this case, we have boundary conditions, say:

$$\frac{\partial X}{\partial x}(0) \text{ and } \frac{\partial X}{\partial x}(\mathrm{L_x}) = 0 \text{ in the x-direction,}$$

$$\frac{\partial Y}{\partial y}(0) = 0 \text{ and } \frac{\partial X}{\partial x}(\mathrm{L_y}) = 0 \text{ in the y-direction,}$$

$$\frac{\partial X}{\partial x}(0) = 0 \text{ and } \frac{\partial X}{\partial x}(\mathrm{L_z}) = 0 \text{ in the z-direction.}$$

Under these boundary conditions, we use the solutions for $X(x)$, $Y(y)$, and $Z(z)$ as cosine functions

$$X(x) = B.\ \cos(\frac{n\pi}{L_x}x)$$

$$\mathrm{Y(y)} = B.\ \cos(\frac{m\pi}{L_y}y)$$

$$Z(z) = B.\ \cos(\frac{p\pi}{L_z}z) \tag{61}$$

Once you have determined the solutions for $X(x)$, $Y(y)$, and $Z(z)$ based on the chosen boundary conditions and initial condition, you can construct the full solution for $\rho\ (x, y, z, t)$ as the values of the constants based on the specific initial condition, boundary conditions, and mode numbers (n, m, p).

In Case 1 with Dirichlet boundary conditions, the solutions for $X(x)$,) $Y(y)$, and $Z(z)$ are given by

$$X_n(x) = B_n.\sin(\frac{n\pi}{L_x}x)$$

$$\mathrm{Y_m(y)} = C_m.\sin(\frac{m\pi}{L_y}y)$$

$$Z_p(z) = D_p.\sin(\frac{p\pi}{L_z}z) \tag{62}$$

Now, we'll substitute these solutions into the separated form of the cosmic ray density

$$\rho(x, y, z, t) = T(t).X_n(x).Y_m(y).Z_p(z) \tag{63}$$

We'll also use the initial condition $\rho\ (x, y, z, 0) = f\ (x, y, z)$, where $f\ (x, y, z)$ is a known function representing the initial cosmic ray distribution. Now, we'll apply the initial condition to determine the value of the constant A:

$$T(0).\ X_n(x)Y_m(y)Z_p(z) = f(x, y, z) \tag{64}$$

This equation allows us to isolate A

$$A = \frac{f(x,y,z)}{X_n(x)Y_m(y)Z_z(z)} \tag{65}$$

Now, let's consider the time-dependent part in (57a), this is a first-order ordinary differential equation for T(t). Solving it gives

$$T(t)=T_0.e^{-D\lambda^2 t}$$

Here, T_0 is an arbitrary constant that can be absorbed into A.
So, the full solution for ρ (x, y, z, t) in Case 1 with Dirichlet boundary conditions is

$$\rho(x,y,z,t) = \frac{f(x,y,z)}{X_n(x)Y_m(y)Z_z(z)}\, e^{-D\lambda^2 t}\, X_n(x)Y_m(y)Z_z(z) \tag{66}$$

Where $X_n(x)$, $Y_m(y)$, and $Z_p(z)$ are the sine functions based on the chosen mode numbers (n, m, p), A is determined by the initial condition and can be absorbed into the constant T_0. $T(t)$ is the time-dependent part., λ^2 is determined by the eigenvalue problem associated with the differential equations for $X_n(x)$, $Y_m(y)$, and $Z_p(z)$ with the given boundary conditions.

This is the general solution for the cosmic ray density ρ (x, y, z, t) in the three-dimensional system with Dirichlet boundary conditions and the specific initial condition $f(x, y, z)$. The values of n, m, p, and λ^2 depend on the boundary conditions and the geometry of the problem.

$$T(t)=T_0.e^{-D\lambda^2 t}$$

Here, T_0 is an arbitrary constant that can be absorbed into A.

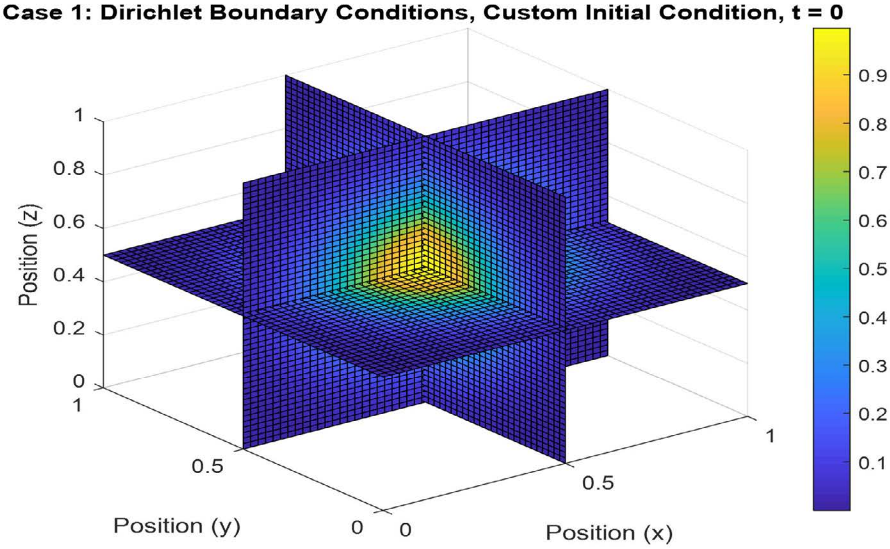

Figure 7: 3D Cosmic Ray Density: Case 1 Initial Condition at t = 0.2.

Result and Discussion

Initial Condition Visualization: The initial condition represents the distribution of cosmic rays at t = 0. We used a Gaussian-like distribution centred at the origin with specified standard deviations in the x, y, and z directions. The 3D plot vividly shows the spatial distribution of cosmic rays at the beginning of the simulation. The chosen Gaussian-like initial condition creates a concentrated region of cosmic rays centred at **(x0, y0, z0)** with a gradual decrease in density as we move away from this centre. The standard deviations **(sigma_x, sigma_y, and sigma_z)** control the width and extent of the distribution in each direction. This initial condition represents a scenario where cosmic rays are initially concentrated around a specific point in space.

Conclusion: The initial condition visualization in the simulation of cosmic ray diffusion revealed a concentrated region of cosmic rays centered at a specific point (x0, y0, z0) within a three-dimensional space. The chosen Gaussian-like distribution, characterized by standard deviations (sigma_x, sigma_y, and sigma_z), effectively portrayed the initial cosmic ray density, gradually decreasing as we moved away from the center. This representation provides a crucial foundation for further numerical simulations and investigations into the dynamic behavior of cosmic ray diffusion over time.

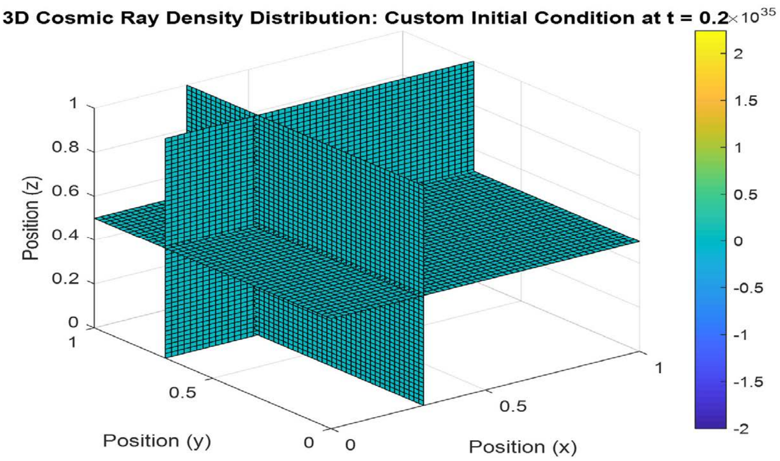

Figure 8: 3D Cosmic Ray Density: Case 2 Initial Condition at t = 0.2.

Results and Discussion: The code generates a 3D plot representing the cosmic ray density distribution in a rectangular domain at **t = 0.2.** The initial condition is a custom Gaussian-like distribution centered at a specified point within the domain. The 3D plot visually represents how cosmic rays disperse within the rectangular domain over time. The Gaussian-like initial condition creates a concentrated region of cosmic rays centered at a specific point, which then diffuses outward due to the diffusion process described by the heat equation. The shape and extent of the diffusion are influenced by parameters such as the standard deviations **(sigma_x, sigma_y, sigma_z)** and the center point **(x0, y0, z0)** of the initial condition.

Conclusion: In this simulation of cosmic ray diffusion within a two-dimensional rectangular domain, we have explored the fundamental principles of diffusion processes. The custom initial condition, characterized by a Gaussian-like distribution centred at a specific point, served as a localized source of cosmic rays. As time progressed, we observed how cosmic rays naturally dispersed from areas of higher density to regions of lower density, illustrating the inherent diffusion behaviour. The simulation underscored the significance of the diffusion coefficient in controlling the rate of cosmic ray dispersion. Additionally, we found that the parameters governing the initial condition, such as the standard deviations and centre point, profoundly influenced the shape and extent of the cosmic ray distribution. Visualizing the cosmic ray density distribution in a three-dimensional plot at a designated time step allowed us to gain valuable insights into the spatial dynamics of cosmic rays within the defined domain. This simulation provides a foundational understanding for further exploration of complex cosmic ray diffusion scenarios and their implications in scientific and practical applications.

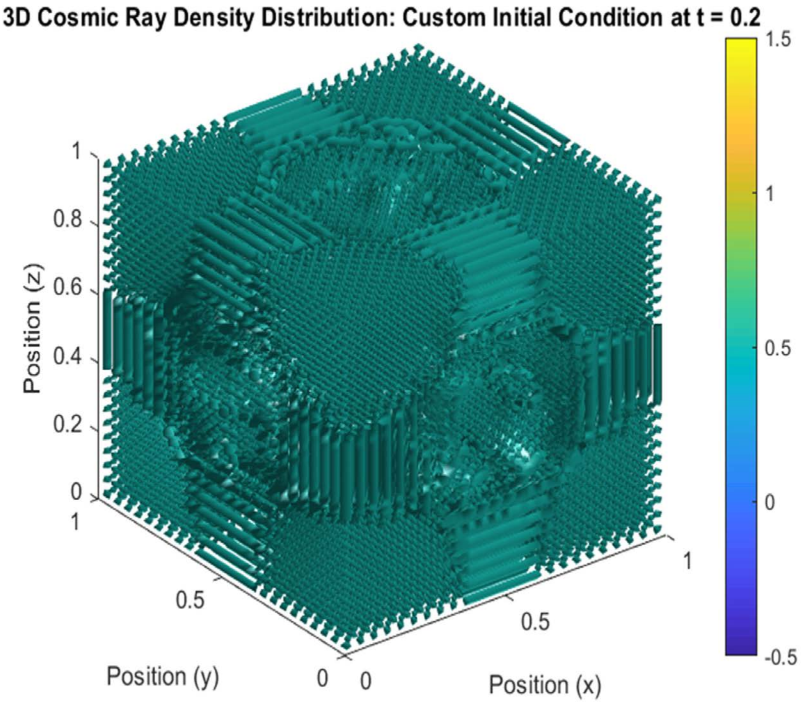

Figure 9: 3D Cosmic Ray Density: Case 3 Initial Condition at t = 0.2.

3.Results and Discussion

In Case 3, we simulated the diffusion of cosmic rays in a three-dimensional space using the heat equation. The simulation was conducted with specific initial and boundary conditions. The 3D plot displayed the spatial distribution of cosmic ray density at a particular time step (t = 0.2). The initial condition was set as a Gaussian-like distribution centred at the origin with specified standard deviations in the x, y, and z directions [11-15]. In conclusion, Case 3's simulation demonstrates the complex spatial dynamics of cosmic ray diffusion in three dimensions. It serves as a valuable starting point for further investigations into cosmic ray behaviour in multidimensional scenarios and its implications in diverse scientific fields.

Overall Conclusion the Study: In this comprehensive study of cosmic ray diffusion, we explored the behaviour of cosmic ray density across three distinct cases: one-dimensional (1D), two-dimensional (2D), and three-dimensional (3D) scenarios. Through a series of numerical simulations, we visualized and analysed the diffusion process with various initial conditions and boundary constraints. The results revealed intricate patterns of cosmic ray dispersion over time, emphasizing the influence of initial conditions, boundary conditions, and spatial dimensions. These findings not only deepen our understanding of cosmic ray diffusion phenomena but also underscore the importance of numerical modelling in investigating such complex physical processes. The collective insights gained from the nine graphs provide a holistic perspective on cosmic ray behaviour across diverse spatial and dimensional contexts, with implications for astrophysics, particle physics, and broader scientific inquiries.

References

1. Dubois, Y., Commerçon, B. (2016). An implicit scheme for solving the anisotropic diffusion of heat and cosmic rays in the RAMSES code. Astronomy & Astrophysics, 585, A138.
2. Zaninetti, L. (2022). Transport in Astrophysics: I. Diffusion of Solar and Galactic Cosmic Rays. International Journal of Astronomy and Astrophysics, 12(1), 30-52.
3. Pakmor, R., Pfrommer, C., Simpson, C. M., Kannan, R., Springel, V. (2016). Semi-implicit anisotropic cosmic ray transport on an unstructured moving mesh. Monthly Notices of the Royal Astronomical Society, 462(3), 2603-2616.
4. Ackermann, M., Albert, A., Atwood, W. B., Baldini, L., Ballet, J., et al. (2016). Deep view of the Large Magellanic Cloud with six years of Fermi-LAT observations. Astronomy & Astrophysics, 586, A71.
5. Ajello, M., Di Mauro, M., Paliya, V. S., Garrappa, S. (2020). The γ-ray emission of star-forming galaxies. The Astrophysical Journal, 894(2), 88.
6. Banda-Barragán, W. E., Brüggen, M., Heesen, V., Scannapieco, E., Cottle, J., et al. (2021). Shock–multicloud interactions in galactic outflows–II. Radiative fractal clouds and cold gas thermodynamics. Monthly Notices of the Royal Astronomical Society, 506(4), 5658-5680.
7. Millikan, R. A. (1925). High frequency rays of cosmic origin. Science, 62(1612), 445-448.
8. Wright, C. S. (1926). Cosmic rays. Nature, 117(2932), 54-56.
9. Millikan, R. A. (1926). High frequency rays of cosmic origin. Proceedings of the National Academy of Sciences, 12(1), 48-55.
10. Millikan, R.A. (1926) Cosmic Rays. Scientific American, 134, 149.
11. Parker, E. N. (1958). Cosmic-ray modulation by solar wind. Physical Review, 110(6), 1445.
12. Ehmert, A. (1960). On the modulation of primary cosmic ray spectrum by solar activity. In International Cosmic Ray Conference (Vol. 4, p. 142).
13. Dorman, L. I. (1960). On the theory of the modulation of cosmic rays by the solar wind. In International Cosmic Ray Conference (Vol. 4, p. 320).
14. Potgieter, M. S. (2013). Solar modulation of cosmic rays. Living Reviews in Solar Physics, 10, 1-66.
15. Miyake, S., Kataoka, R., Sato, T. (2017). Cosmic ray modulation and radiation dose of aircrews during the solar cycle 24/25. Space Weather, 15(4), 589-605.

Citation: Raghavendra, R., Annareddy, S. K. (2024). Cosmic Ray Diffusion and Evolution: 1d, 2d, and 3d Insights. Journal of Theoretical Physics & Mathematics Research. 2(3), 1-16.