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Bifurcation Analysis and Model Predictive the Control of the Reaction-Diffusion Fire Dynamic Model

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Abstract

The many wildfire disasters that have caused tremendous destruction make it necessary to understand how the intensity of the fire varies with time and the path taken by the flames as they spread through a region. To gain a scientific understanding of these two issues and to be able to control the wildfire spread it is important to develop computational strategies to predict the paths taken by the flames and to minimize the damage done. In this paper, bifurcation analysis and multiobjective nonlinear model predictive control calculations are conducted on the reaction-diffusion wild spread model. Bifurcation analysis and multiobjective nonlinear model predictive control calculations of the reaction-diffusion fire dynamic model was performed. Bifurcation analysis was performed using the MATLAB software MATCONT. The multiobjective nonlinear model predictive control was performed with the optimization Language PYOMO. The bifurcation analysis reveals the existence of limit points and the multiobjective nonlinear model predictive control calculations demonstrate that the rate at which the area is being burned could be minimized and the rate at which the fire is being extinguished could be maximized simultaneously.

Keywords: Optimal Control, Bifurcation, Fire Model

1. Introduction

Bush and McLaughlin introduced the subject of fire science [1]. Zeldovich et al developed a mathematical theory of combustion and explosions [2]. Platt modeled fire spread using a time-based probability approach while Ramachandran, developed a nondeterministic model of fire spread [3,4]. Frieman, looked at various computer Models for fire and smoke [5]. Madrzykowski,et al., developed a sprinkler fire suppression algorithm for the GSA engineering fire assessment system [6]. Takeda et al., developed simplified fire growth models for risk-cost assessment In apartment buildings Tat and Hasofer, ‘modeled the spread of fire using non-stationary stochastic processes [7,8].

Honecker, and Peschel, introduced length scales and power laws into two dimensional forest-fire models [9]. Medez et al., used hyperbolic reaction-diffusion equations for forest fire models [10]. Tat developed stochastic models and performed optimal Control of Compartment Fires [11]. Ashwin,et al., described the existence of traveling fronts for the KPP equation(a reaction-diffusion equation used in fire modeling) with spatiotemporal delay [12].

Zhong et al., developed a statistical analysis of the current status of China forest fire safety Dercole and Maggi discovered a border collision bifurcation in a forest fire model. Maggi and Rinaldi produced a second-order impact model for forest fire regimes [13-15]. Ferragut et al., developed a numerical method for solving convection-reaction-diffusion multivalued equations in fire spread modeling [16]. Hollis et al., tested woody fuel consumption models for application in Australian southern eucalypt forest fires [17]. Grishin and Filkov, developed a deterministic-probabilistic system for predicting forest fire hazards [18]. Min et al., studied the dynamic properties of a forest fire model that involved reaction-diffusion equations [19].

1.1. Objectives of this Work

Most of the theoretical work about fire models involves bifurcation analysis and single-objective optimal control tasks performed disjointly. In this work, both bifurcation analysis and multiobjective nonlinear model predictive control calculations were performed on the reaction-diffusion dynamic fire model [19]. The bifurcation analysis revealed the existence of limit points and the MNLMPC calculations resulted in the Utopia solution. This confirms the result obtained by Sridhar that when MNLMPC calculations were performed on problems that exhibited Limit and Branch points the Utopia point was always obtained [20]. The paper is organized as follows. First, the reaction-diffusion fire dynamic model is described. This is followed by the numerical procedures for the bifurcation analysis and multiobjective nonlinear model predictive control (MNLMPC). The results are then described and discussed followed by the conclusions.

1.2. Model Description

The reaction-diffusion fire dynamic model that is used consists of the equations

$$\frac{\partial u}{\partial t} = d_1 \frac{\partial^2 u}{\partial x^2} - au^2 - \frac{uv}{b(u+1)}$$
$$\frac{\partial v}{\partial t} = d_2 \frac{\partial^2 v}{\partial x^2} - cv + \frac{uv}{b(u+1)} \quad (1)$$

In this equation, u is the area of the burned forest and v represents the area where the fire has been put out. x is the space coordinate and while t is the time. $\frac{\partial u}{\partial t};\frac{\partial v}{\partial t}$ represents the rate at which the area is being burned and the rate at which the area of the fire is being extinguished. $\frac{\partial^2 u}{\partial x^2},\frac{\partial^2 v}{\partial x^2}$ are the diffusion terms of u and v in space. d_1, d_2 represent the diffusion coefficients and a, b, and c are model parameters. The parameter values are a = 0.0588 and c = 4. d_1, d_2 are 0.05 and 3. b is the bifurcation and control variable.

1.3. Bifurcation Analysis

Multiple steady states and oscillatory behavior occur in various situations. Multiple steady states occur because of Branch and Limit bifurcation points cause multiple steady states. Hopf bifurcation points produce oscillatory behavior. Limit cycles. The MATLAB program MATCONT [21, 22]. Is commonly used software to locate limit points, branch points, and Hopf bifurcation points? Consider an ODE system

$$\dot{x} = f(x, \beta) \quad (2)$$

$x \in R^n$. Defining the matrix A as

$$A = \begin{bmatrix} \frac{\partial f_1}{\partial x_1} & \frac{\partial f_1}{\partial x_2} & \frac{\partial f_1}{\partial x_3} & \frac{\partial f_1}{\partial x_4} \cdots\cdots & \frac{\partial f_1}{\partial x_n} & \frac{\partial f_1}{\partial \beta} \\ \frac{\partial f_2}{\partial x_1} & \frac{\partial f_2}{\partial x_2} & \frac{\partial f_2}{\partial x_3} & \frac{\partial f_2}{\partial x_4} \cdots\cdots & \frac{\partial f_2}{\partial x_n} & \frac{\partial f_2}{\partial \beta} \\ \cdots & & & & & \\ \cdots & & & & & \\ \frac{\partial f_n}{\partial x_1} & \frac{\partial f_n}{\partial x_2} & \frac{\partial f_n}{\partial x_3} & \frac{\partial f_n}{\partial x_4} \cdots\cdots & \frac{\partial f_n}{\partial x_n} & \frac{\partial f_n}{\partial \beta} \end{bmatrix} \quad (3)$$

β Is the bifurcation parameter. The matrix A can be written in a compact form as

$$A = [B \mid \quad \partial f / \partial \beta] \quad (4)$$

The tangent at any point x; ($v = [v_1, v_2, v_3, v_4, \ldots v_{n+1}]$) must satisfy

$$Av = 0 \quad (5)$$

The matrix B must be singular at both limit and branch points.. The n+1 th component of the tangent vector V_{n+1} = 0 at a limit point (LP) and for a branch point (BP) the matrix $\begin{bmatrix} A \\ v^T \end{bmatrix}$ must be singular. At a Hopf bifurcation,

$$\det(2 f_x(x, \beta) @ I_n) = 0 \quad (6)$$

@ indicates the bialternate product while I_n is the n-square identity matrix. Hopf bifurcations cause unwanted oscillatory behavior and should be eliminated because oscillations make optimization and control tasks very difficult. More details can be found in Kuznetsov and Govaerts [23-25].

1.4. Multi Objective Nonlinear Model Predictive Control Algorithm

Flores Tlacuahuaz first proposed the Multiobjective nonlinear model predictive control method [26]. that does not involve weighting functions, nor does it impose additional constraints on the problem unlike the weighted function or the epsilon correction method [27]. For a a set of ODE

$$\frac{dx}{dt} = F(x, u)$$
$$h(x, u) \le 0 \quad x^L \le x \le x^U; \quad u^L \le u \le u^U \quad (7)$$

let $\sum_{t_i=0}^{t_i=t_f} p_j(t_i)$ (j=12..n); be the variables that need to be minimized/maximized simultaneously, t_f being the final time value, and n the total number of variables that need to be optimized simultaneously. In this MNLMPC method dynamic optimization problems that independently minimize/maximize each variable $\sum_{t_i=0}^{t_i=t_f} p_j(t_i)$ are solved individually. The minimization/maximization of each $\sum_{t_i=0}^{t_i=t_f} p_j(t_i)$ will lead to the values p_j^*. Then the optimization problem that will be solved is

$$\min\left(\sum_{j=1}^{n}\left(\sum_{t_i=0}^{t_i=t_f} p_j(t_i) - p_j^*\right)\right)^2$$
$$subject\ to\ \frac{dx}{dt} = F(x, u); \quad h(x, u) \le 0 \quad (8)$$
$$x^L \le x \le x^U; \quad u^L \le u \le u^U$$

This will provide the control values for various times. The first obtained control value is implemented and the rest are ignored. The procedure is repeated until the implemented and the first obtained control values are the same or if the Utopia point ($\sum_{t_i=0}^{t_i=t_f} p_j(t_i) = p_j^*$; for all j) is achieved. The optimization package in Python, where the differential equations are automatically converted to algebraic equations will be used. The resulting optimization problem was solved using IPOPT [28,29]. The obtained solution is confirmed as a global solution with BARON [30]. To summarize the steps of the algorithm are as follows

- p_j^* Minimize/maximize $\sum_{t_i=0}^{t_i=t_f} p_j(t_i)$. This will lead to the value at various time intervals ti. The subscript i is the index for each time step.
- Minimize $\sum_{j=1}^{n}(\sum_{t_i=0}^{t_i=t_f} p_j(t_i) - p_j^*))^2$. This will provide the control values for various times.

• Implement the first obtained control values and discard the remaining.

• The steps are repeated until there is an insignificant difference between the implemented and the first obtained value of the control variables or if the Utopia point is achieved.

2. Results and Discussion

Bifurcation and Multiobjective nonlinear model predictive control calculations were performed on the reaction-diffusion fire model.

Bifurcation calculations were performed using the MATLAB program MATCONT. Here the second-order derivatives were discretized using (N=10) elements. The MATLAB code for this would be:

```
d2u(i)= (u(i-1)-2*u(i)+u(i+1))/ (h*h)
d2v(i) = (v(i-1)-2*v(i)+v(i+1))/ (h*h)
```

where h is (1/N+1) and i is the number of the element. d2u , d2v represent $\frac{\partial^2 u}{\partial x^2}, \frac{\partial^2 v}{\partial x^2}$. This would convert the right-hand sides of Eq. 1 into algebraic equations. After discretization, the equations would resemble Eq. 2 and MATCONT is used. Bifurcation analysis revealed the existence of two limit points given by

$(u_i\, v_i\, b)$(for i = 1,2,...10) as

(1.096486 1.187844 1.265469 1.321880 1.351552 1.351552 1.321880 1.265469 1.187844 1.096486 0.986917 0.976546 0.968824 0.963702 0.961149 0.961149 0.963702 0.968824 0.976546 0.986917 0.518885)
and
(1.786953 2.523190 3.140729 3.583822 3.814724 3.814724 3.583822 3.140729 2.523190 1.786953 0.988814 0.980074 0.973620 0.969360 0.967241 0.967241 0.969360 0.973620 0.980074 0.988814 0.490588). These limit points are shown in Figure 1.

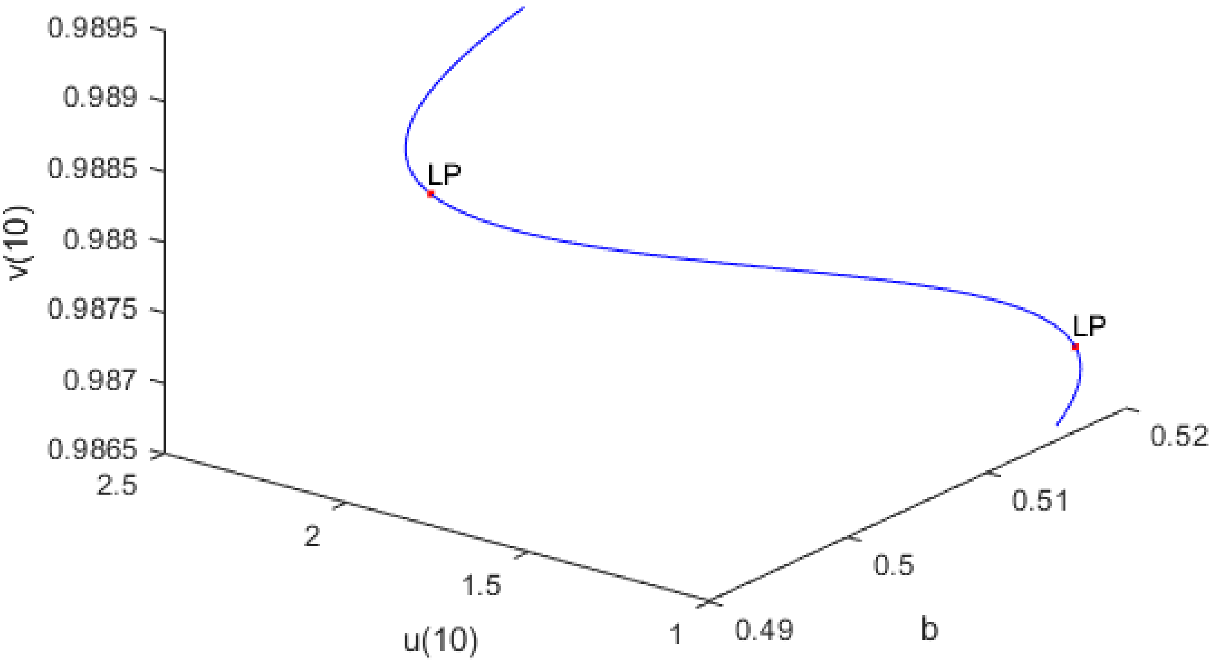

Figure 1: Limit Points when Bifurcation Analysis was Performed on Reaction-Diffusion Fire Model

For the MNLMPC calculations $\sum_{t_i=0}^{t_i=t_f} (\frac{\partial u(t_i)}{dt})_j$ was minimized and $\sum_{t_i=0}^{t_i=t_f} (\frac{\partial u(t_i)}{dt})_j$ was maximized. The minimization of $\sum_{t_i=0}^{t_i=t_f} (\frac{\partial u(t_i)}{dt})_j$ resulted in a value of 0 while the maximization of $\sum_{t_i=0}^{t_i=t_f} (\frac{\partial u(t_i)}{dt})_j$ resulted in a value of 400. The multiobjective optimal control problem that was then solved resulted in the minimization of $(\sum_{t_i=0}^{t_i=t_f} (\frac{\partial v(t_i)}{dt})_j - 400)^2 + (\sum_{t_i=0}^{t_i=t_f} (\frac{\partial u(t_i)}{dt})_j - 0)^2$ subject to Eq. 1. He second derivative was obtained using the PYOMO command.

```
m.d2udx2 = DerivativeVar(m.u, wrt=(m.x, m.x))
m.d2vdx2 = DerivativeVar(m.v, wrt=(m.x, m.x))
```

This MOOC resulted in a Utopia point of 0. The first obtained control value of b was implemented and the rest were discarded. This procedure was repeated until there was no difference between the first and second control values of b. This MNLMPC value of b obtained was 0.9416. Fig. 2 shows the b vs t profile for the MNLMPC calculation while figs 3 and 4 show the (u x t) and (v x t) surfaces.

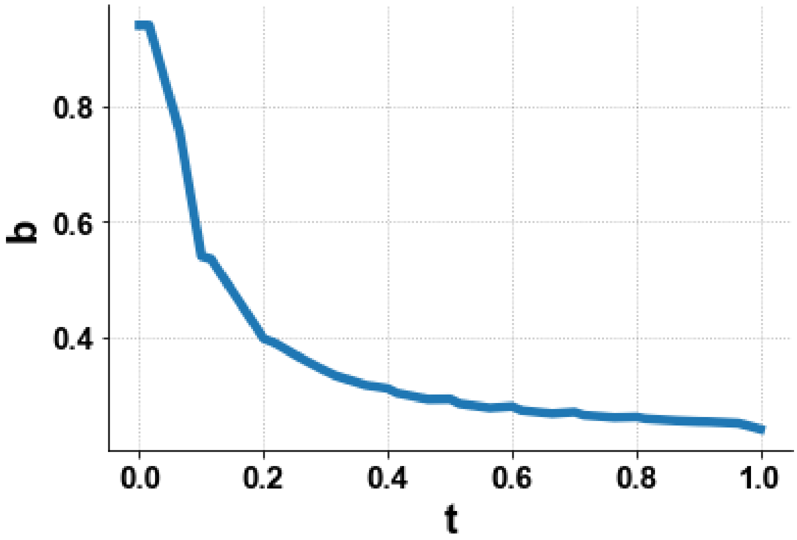

Figure 2: b vs t for the MNLMPC Calculation

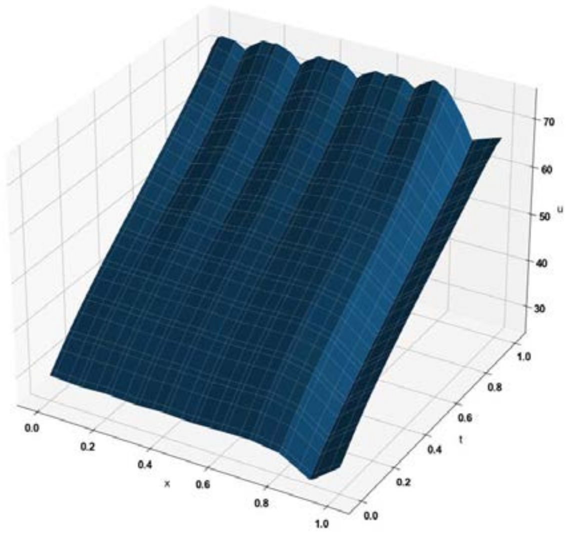

Figure 3: (u, x, t) Surface for the MNLMPC Calculation

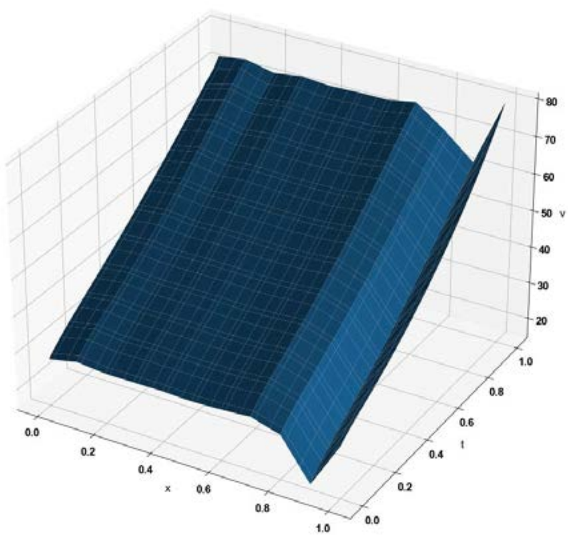

Figure 4: v, x, t Surface for the Mnlmpc Calculation

A recently published article by Sridhar(2024) demonstrated that when MNLMPC calculations were performed on problems that exhibited Limit and Branch points the Utopia point was always obtained. This was done by incorporating the singularity condition (because of the limit and branch points) on the co-state equation for the optimal control problem. This result confirms the theorem of Sridhar(2024).,

3. Conclusions

Bifurcation analysis and multiobjective nonlinear model predictive control were performed on the reaction. Diffusion fire spread model. The bifurcation analysis revealed the existence of limit points which imply that the fire spread direction could change significantly. The multiobjective nonlinear model predictive control resulted in the Utopia solution. This demonstrates that the rate at which the area is being burned could be minimized and the rate at which the fire is being extinguished could be maximized effectively and simultaneously.

Data Availability Statement

All data used is presented in the paper

Conflict of Interest

The author, Dr. Lakshmi N Sridhar has no conflict of interest.

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Author Contribution

Dr. Sridhar is the sole author and did all the work for this paper

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